A HIERARCHY OF DYNAMIC STATES OF RELAXATION

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Abstract

We define a hierarchy of dynamic relaxed gas spheres as solutions of the Poisson equation coupled to a hierarchy of approximations of the Liouville equation leading, when this equation is satisfied, to the well-known isothermal gas spheres in the static case, but also to a new form of dynamic maximum relaxation. The two previous steps of the hierarchy correspond to an increasing degree of local relaxation at the center of the configuration.

1   Introduction

Embodied into the concept of equilibrium there are two ingredients. The first and most primitive is the idea of stillness. The second one is the idea that equilibrium is the latest stage of a macroscopic process to which a system has been driven by a microscopic mechanism at work in every cell of the system. It would be superfluous to distinguish these two ingredients if we assumed that the process under consideration necessarily leads to stillness, but we do not need to do that. Here we distinguish between the concept of equilibrium proper with the usual connotation of stillness attached to it, and the concept of maximum relaxation which is a concept that describes some terminal dynamic stages of evolution.

We have already considered this distinction before[1] in the framework of General relativity and more precisely in the description of the matter contend of some Robertson-Walker models of universe using Relativistic kinetic theory. Since the Universe is not stationary, i.e, there does not exist any global time-like Killing vector-field, but expanding it is not possible to assume that the cosmological fluid is in equilibrium in the usual sense. But it is possible to imagine that this fluid is in an state of maximum relaxation, which in the quoted reference we called generalized equilibrium. We believe now that the

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[1]Ref. [1]

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concept of maximum relaxation may be useful also in more classical contexts, either to describe laboratory systems with time-dependent boundary conditions or external perturbations, or in astrophysical descriptions of self-gravitating systems in the strict framework of Newtonian gravity. This is more specifically the problem that we discuss in this paper. We hope that this restriction to a more classical problem may help to uncover the usefulness of the concept of maximum relaxation.

Actually the description in a particular case of an state of maximum relaxation is not the unique goal that we pursue in this paper. We are equally interested in describing some intermediate steps that may settled down in systems which do not reach uniformly maximum relaxation. It might well be that some of these partial stages of relaxation become interesting by themselves.

It is possible to distinguish three domains in the physics of the kinetic theory of gases: i) the physics of equilibrium, ii) the physics near equilibrium where perturbation theory can be used, and iii) the jungle where the most difficult problems arise. This paper is a modest contribution to a partial clarification of a small territory of this non perturbative jungle.

2 Particular field integrals

Let us consider a spherical mass distribution with Newtonian gravitational potential $V(t, r)$ and a massive particle free-falling in the corresponding gravitational field according to the Newtonian equations of motion:

$$\frac{dv^i}{dt} = -\delta^{ij}\partial_j V(t, r), \quad v^i = \frac{dx^i}{dt}, \quad r = \sqrt{\epsilon_i(x^i)^2}, \quad \epsilon_i = 1 \tag{1}$$

We shall say that the following quadratic expression:

$$I = \frac{1}{2}A_{ij}v^i v^j + B_i v^i + C \tag{2}$$

is a spherically symmetric *Field integral* if:

- A_{ij} and B_i have the following tensor structure:

$$A_{ij} = M\delta_{ij} + Nn_i n_j, \quad B_i = Pn_i \tag{3}$$

  with:

$$n_i = \delta_{ij}n^j, \quad n^i = r^{-1}x^i, \tag{4}$$

  M, N, P and C being functions of t and r.

- The total derivative of I with respect to t is independent of the velocity:

$$\frac{dI}{dt} = F(t, r) \tag{5}$$

  where F is a function that we shall call the source of the field integral.

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Requiring I to satisfy this last equation taking into account 2, yields, to start with, the following set of conditions:

$$\partial_{(k} A_{ij)} = 0 \tag{6}$$

$$\frac{1}{2}\partial_t A_{ij} + \partial_{(i} B_{j)} = 0 \tag{7}$$

$$-\delta^{jk}\partial_k V A_{ij} + \partial_t B_i + \partial_i C = 0 \tag{8}$$

$$-\delta^{jk}\partial_j V B_k + \partial_t C = F \tag{9}$$

And taking into account 3 we obtain the following system of equations:

$$M' + \frac{2N}{r} = 0, \quad N' - \frac{2N}{r} = 0 \tag{10}$$

$$\dot{M} + \frac{2P}{r} = 0, \quad \dot{N} + 2P' - \frac{2P}{r} = 0 \tag{11}$$

$$-V'(M + N) + \dot{P} + C' = 0 \tag{12}$$

$$-V'P + \dot{C} = F \tag{13}$$

where a dot means partial derivative with respect to t and a prime a partial derivative with respect to r.

Integrating the system of Eqs. 10-12 we get:

$$N = -\nu r^2, \quad M = \nu r^2 + \mu \tag{14}$$

$$P = -\frac{1}{2}\dot{\mu}r \tag{15}$$

$$C = \mu V + \frac{1}{4}\ddot{\mu}r^2 + \sigma \tag{16}$$

where ν is a constant, and $\mu(t)$ and $\sigma(t)$ are two functions of t. Taking into account 15 and 16 Eq. 13 becomes:

$$\dot{\mu}(V + \frac{1}{2}V'r) + \mu\dot{V} + \frac{1}{4}\mu^{(3)}r^2 + \dot{\sigma} = F \tag{17}$$

where $\dot{\mu}^{(3)}$ means the third derivative of μ with respect to t.

The concept of a field integral is an obvious generalization of the concept of constant of motion to which it reduces if $F = 0$. This concept can be useful only if some other considerations restrict its source F. This will be the case in the sequel of this paper.

Collecting the above results, I can be written as:

$$I = E + \frac{1}{2}\nu J^2 \tag{18}$$

where:

$$E = \mu(\frac{1}{2}\vec{v}^2 + V) - \frac{1}{2}\dot{\mu}r\vec{v}\vec{n} + \frac{1}{4}\ddot{\mu}r^2 + \sigma, \quad J^2 = r^2[\vec{v}^2 - (\vec{v}\vec{n})^2]. \tag{19}$$

J, which is the angular momentum per unit mass with respect to the center of the mass distribution, is a constant of the motion, but E, which is proportional to a generalization of the energy per unit mass to which it reduces if $\mu = 1$ and $\dot{V} = 0$ is a field integral:

$$\frac{dE}{dt} = F, \quad \frac{dJ}{dt} = 0 \tag{20}$$

3 Coupling gravitation to a kinetic equation

We assume in this paper that the density $\rho(t, x^i)$ of the gas under consideration is given as the mean value of a kinetic distribution function $f(t, x^i, v^j)$. More precisely we assume that the density ρ in the Poisson equation

$$\triangle V = \rho \tag{21}$$

where units have been chosen such that:

$$4\pi G = 1 \tag{22}$$

is given by:

$$\rho(t, x^i) = \int f(t, x^i, v^j)\, dv^1 dv^2 dv^3 \tag{23}$$

where $f \geq 0$ is a solution of a kinetic equation:

$$\partial_t f + v^i \partial_i f - \delta^{ij}\partial_i V \frac{\partial f}{\partial v^j} = B \tag{24}$$

with $B(t, x^i)$ being a function independent of the velocity that has to be zero when equilibrium or maximum relaxation is reached.

Since we are interested only in spherically symmetric configurations, B, ρ and V will be functions of t and r, the equation 21 becoming:

$$V'' + \frac{2}{r}V' = \rho, \tag{25}$$

and f will be a function of t, r, $\vec{v}^2$ and $\vec{v}\vec{n}$.

Finally, we shall assume that the domain on which 23 and 24 apply includes the center of the gas sphere and is bounded by a sphere of finite radius R. And also that in this domain the density and the mass M are finite. Consistently with these assumptions we shall assume without any loss of generality that

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$V(t, 0) = 0$, but we have to keep in mind that the value of this quantity should be chosen appropriately to match the interior solution to the solution corresponding to a next layer beyond the surface $r = R$. Complete models and the matching problems are not considered in this paper which deals only with the central cores of gas spheres where presumably some degree of relaxation has already been achieved.

4 The road to equilibrium or maximum dynamic relaxation

We assume from now on that f is a distribution function that has the following form:

$$f = m \exp(-I) \tag{26}$$

where m is the mass of the particles of the gas and I is a field integral of the type that we considered in section 2, with N, M, P and C as in 14, 15 and 16, and with F being given by Eq. 17. Therefore the r-h-s of 24 is:

$$B = -fF \tag{27}$$

Evaluating the integral 23 we obtain:

$$\rho = (2\pi)^{3/2} m \frac{\exp(-\mu V - \sigma - \frac{1}{4}\ddot{\mu}r^2 + \frac{1}{8}\mu^{-1}\dot{\mu}^2 r^2)}{\mu^{3/2}(1 + \mu^{-1}\nu r^2)} \tag{28}$$

We follow below the hierarchy of steps leading to equilibrium or to maximum relaxation described by an ordered set of restrictive assumptions on the function $F(t, r)$. We shall consider a power expansion of the this function with respect to the variable r:

$$F(t, r) = F_0(t) + F_1(t)r + \cdots + \frac{1}{n!}F_n(t)r^n + \cdots \tag{29}$$

and we shall examine the models which are obtained assuming that more and more terms, starting with F_0, of this expansion are zero.

Concomitantly with 29 we shall consider the power series expansions of $V(t, r)$ and $\rho(t, r)$ derived from 28:

$$V(t, r) = V_0(t) + V_1(t)r + \cdots + \frac{1}{n!}V_n(t)r^n + \cdots \tag{30}$$

$$\rho(t, r) = \rho_0(t) + \rho_1(t)r + \cdots + \frac{1}{n!}\rho_n(t)r^n + \cdots \tag{31}$$

Integrating 25 term by term we obtain:

$$V_1 = 0, \quad V_n = \frac{n-1}{n+1}\rho_{n-2}, \quad n > 1 \tag{32}$$

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the first of these equalities coming from the assumption that the density is finite at the center.

First step. Assuming that $F_0(t) = 0$ in Eq. 17 yields:

$$\dot{\mu}V_0 + \mu\dot{V}_0 + \dot{\sigma} = 0, \quad \text{or} \quad \mu V_0 + \sigma = Constant_1, \tag{33}$$

and therefore using 28 and 32 we can write:

$$V_2 = \frac{1}{3}\rho_0 = a\mu^{-3/2}, \quad \text{with} \quad a = Constant_2. \tag{34}$$

From $V_1 = 0$ it follows that $F_1 = 0$. Also from 28 and $V_1 = 0$ it follows that $\rho_1 = 0$ and therefore from 32 we have also $V_3 = 0$.

Second step. The next step is then to assume also that $F_2 = 0$, i.e.:

$$\dot{\mu}V_2 + \frac{1}{2}\mu\dot{V}_2 + \frac{1}{4}\dot{\mu}^{(3)} = 0 \tag{35}$$

and using 34 we get:

$$\dot{\mu}^{(3)} + a\mu^{-3/2}\dot{\mu} = 0 \tag{36}$$

This is the first relevant result of this paper as it gives the equation that governs the dynamics of the central core of the spherical gas configuration.

This equation has two types of solutions:

- A. Those with $\dot{\mu} = 0$

- B. Those with $\dot{\mu}$ non identically zero

Type A. In the first case Eq. 17 becomes:

$$\mu\dot{V} + \dot{\sigma} = F \tag{37}$$

therefore from 33 it follows that $F_0 = 0$. In fact, using repeatedly Eqs. 28, 32 and 37 it is easy to prove that $F = 0$, meaning that f is a solution of the Liouville equation, and that $\dot{\rho} = 0$ and $\dot{V}_n = 0$ for $n > 0$. The potential $V(r)$ will be obtained integrating the non-linear differential equation:

$$V'' + \frac{2}{r}V' = 3a\mu^{-3/2}\frac{\exp(-\mu V)}{1 + \mu^{-1}\nu r^2} \tag{38}$$

If $\nu = 0$ then the configurations thus obtained are the well-known static isothermal gas spheres with temperature:

$$T = \frac{m}{k\mu} \tag{39}$$

where k is Boltzmann's constant.

Type B1 From now on we assume that we are dealing with the case B above, i.e. $\dot{\mu} \neq 0$. In this case the potential $V(t,r)$ is a solution of the following equation:

$$V'' + \frac{2}{r}V' = 3a\mu^{-3/2}\frac{\exp(-\mu V - (\frac{1}{4}\ddot{\mu} - \frac{1}{8}\mu^{-1}\dot{\mu}^2)r^2)}{1 + \mu^{-1}\nu r^2} \tag{40}$$

Type B2 Since from 32 and $\rho_1 = 0$ it follows that $V_3 = 0$ and therefore also $F_3 = 0$ the next step consists in requiring $F_4 = 0$. This yields:

$$3\dot{\mu}V_4 + \mu\dot{V}_4 = 0 \tag{41}$$

From 32 and 28 we have:

$$V_4 = \frac{9}{5}a\mu^{-3/2}(-a\mu^{-1/2} - \frac{1}{2}\ddot{\mu} + \frac{1}{4}\mu^{-1}\dot{\mu}^2 - 2\mu^{-1}\nu), \tag{42}$$

Deriving this expression and substituting in 41 we obtain:

$$\mu^{(3)} + \frac{1}{2}\mu^{-1}\dot{\mu}\ddot{\mu} + (2a\mu^{-3/2} - \frac{1}{4}\mu^{-2}\dot{\mu}^2 + 2\nu\mu^{-2})\dot{\mu} = 0 \tag{43}$$

and using in this equation 36 we get at this step the following equation:

$$\ddot{\mu} = -2a\mu^{-1/2} + \frac{1}{2}\mu^{-1}\dot{\mu}^2 - 4\mu^{-1}\nu \tag{44}$$

Going back to 42 and 32 we see that this implies that:

$$V_4 = 0, \quad \rho_2 = 0 \tag{45}$$

From 28 and the preceding results we have $\rho_3 = 0$ and therefore also $V_5 = 0$ and $F_5 = 0$.

The equation to integrate to get the potential V is:

$$V'' + \frac{2}{r}V' = 3a\mu^{-3/2}\frac{\exp(-\mu V + (\frac{1}{2}a\mu^{-1/2} + \mu^{-1}\nu)r^2)}{1 + \mu^{-1}\nu)r^2} \tag{46}$$

Type B3. This will be the last step. From 28 and 32 we have:

$$V_6 = \frac{180}{7}a\nu^2\mu^{-7/2} \tag{47}$$

Requiring $F_6 = 0$ we obtain:

$$4\dot{\mu}V_6 + \mu\dot{V}_6 = 0 \tag{48}$$

and using 47 we get:

$$a\mu^{-7/2}\dot{\mu}\nu = 0 \tag{49}$$

and since we are considering here only dynamic configurations it follows that:

$$\nu = 0 \tag{50}$$

The evolution equation 44 becomes in this case:

$$\ddot{\mu} + 2a\mu^{-1/2} - \frac{1}{2}\mu^{-1}\dot{\mu}^2 = 0 \tag{51}$$

This is the case of maximum relaxation since, as can easily be proven, the results already obtained imply that:

$$\rho(t) = \rho_0(t) = 3a\mu^{-3/2}, \quad V(t,r) = V_0(t) + \frac{1}{2}a\mu^{-3/2}(t)r^2, \quad F(t,r) = 0 \tag{52}$$

$V_0(t)$ is at this stage an arbitrary function of t to be fixed by matching the core solution to its exterior gravitational field.

5 Explicit solutions

We are in this section interested on the explicit solutions of Eqs. 36, 44 and 51, corresponding to the first and second stages of dynamic partial and total relaxation in the domain where the variable μ is positive.

Eq. 36 has an obvious first integral. Namely:

$$b = \ddot{\mu} - 2a\mu^{-1/2} \tag{53}$$

Considering t as a function of μ an elementary process of integration leads to:

$$\dot{\mu} = \sqrt{2b\mu + 8a\sqrt{\mu} + 2c_1} \tag{54}$$

from where we obtain the general solution of 36:

$$t - t_0 = \int_{\mu(t_0)}^{\mu} \frac{dy}{\sqrt{2by + 8a\sqrt{y} + 2c_1}} \tag{55}$$

the three constants of integration being μ_0, b, and c_1 which are related to $\mu(t_0)$, $\dot{\mu}(t_0)$ and $\ddot{\mu}(t_0)$ by the formulas:

$$b = \ddot{\mu}(t_0) - 2a\mu(t_0)^{-1/2}, \quad c_1 = \frac{1}{2}\dot{\mu}(t_0)^2 - \mu(t_0)\ddot{\mu}(t_0) - 2a\sqrt{\mu(t_0)} \tag{56}$$

The roots of the radical in Eq. 54 are:

$$\sqrt{\mu}_{\pm} = \frac{1}{b}(-2a \pm \sqrt{4a^2 - bc_1}) \tag{57}$$

and the values of $\ddot{\mu}$ for these values are:

$$\ddot{\mu}_{\pm} = \pm \frac{\sqrt{4a^2 - bc_1}}{\sqrt{\mu}_{\pm}}. \tag{58}$$

Therefore if $\sqrt{\mu}_{\pm}$ are real and positive then μ_+ corresponds to a minimum of the function μ and μ_- corresponds to a maximum.

From the preceding considerations it follows that the solutions of 36 can be classified according to the following types:

- Type I1: $c_1 > 0$, $b \geq 0$. The function μ has neither a maximum nor a minimum in the domain of interest $(0 \leq \mu < \infty)$. If the initial value of $\dot{\mu}_0$ is positive μ will increase without limit; otherwise it will reach the value zero.

- Type I2: $c_1 > 0$, $b < 0$. The function μ has a maximum, μ_-.

- Type II1: $c_1 = 0$, $b \geq 0$. The function μ has neither a maximum nor a minimum.

- Type II2: $c_1 = 0$, $b < 0$. The function μ has a maximum, μ_-.

- Type III1: $c_1 < 0$, $b > 0$. The function μ has a minimum, μ_+.

- Type III2: $c_1 < 0$, $b = 0$. The function μ has a minimum, μ_+.

- Type III3: $c_1 < 0$, $b < 0$. The function μ has a maximum, μ_-, a minimum μ_+, and is periodic; the period being:

$$Period = 2 \int_{\mu_+}^{\mu_-} \frac{dy}{\sqrt{2by + 8a\sqrt{y} + 2c_1}} \tag{59}$$

Let us consider the following function of t:

$$S(t) \equiv \ddot{\mu} + 2a\mu^{-1/2} - \frac{1}{2}\mu^{-1}\dot{\mu}^2 + 4\mu^{-1}\nu \tag{60}$$

so that Eq. 44 can be written as $S = 0$. Evaluating the derivative of 60 and using Eq. 36 we readily get:

$$\dot{S} = -\mu^{-1}\dot{\mu}S. \tag{61}$$

This means that $S = 0$ is a conditional first integral of Eq. 36: if a solution $\mu(t)$ of this equation is such that $S(t_0) = 0$ for a particular value of t then it is zero for any time. Equivalently, the solutions of 44 are the solutions 36 for which the initial conditions have been constrained to satisfy 44.

Using 56 and 54 in Eq. 36 yields:

$$c_1 = 4\nu \tag{62}$$

Therefore the general solution of Eq. 44 can be described as the general solution of 36 given above requiring c_1 to satisfy 62. In particular from 50 it follows that the solutions of type II above correspond to configurations of maximum relaxation.

6 Examples

Let us introduce the temperature function $T(t)$ as the following obvious generalization of the temperature parameter 39:

$$T(t) = \frac{m}{k\mu(t)} \tag{63}$$

and let us consider a system of units such that besides 22 one has:

$$\rho_0(t_0) = 3, \quad \mu(t_0) = 1 \tag{64}$$

where t_0 is some particular value of t. The units of time $(\bar{T})$, length $(\bar{L})$ and mass $(\bar{M})$ of this system measured in the MKS system of units are as follows:

$$\bar{T} = (\frac{4}{3}\pi G\rho_0(t_0))^{-1/2}\,\text{s}, \quad \bar{L} = \bar{T}\mu(t_0)^{-1/2}\,\text{m}, \quad \bar{M} = (4\pi G)^{-1}\bar{L}^3\bar{T}^{-2}\,\text{kg} \tag{65}$$

Each set of values of the parameters b, c_1 and the total mass M will define a three parameter family of physical examples depending on the assumed physical values of $\rho_0(t_0)$, $\mu(t_0)$ and M measured in the MKS system of units. Equivalently $T(t_0)$ can be used if we know m. Here we shall assume that m is the mass of an hydrogen atom.

We are going to discuss two particular classes of examples belonging to two different types among those discussed in the preceding section.

Our first example assumes that:

$$b = -2.2, \quad c_1 = -1.8 \tag{66}$$

This corresponds to a type III3 and therefore $\mu(t)$ will be a periodic function. Assuming that t_0 is a time corresponding to a maximum value of μ, then from 34 and 57 it follows that:

$$a = 1, \quad \mu_- = 1, \quad \mu_+ = 0.8 \tag{67}$$

and performing the integral 59 we obtain:

$$Period = 5.4 \tag{68}$$

From 25 we have that the radius R and the mass M of the gas configuration at any time are related by the formula:

$$M = 4\pi R^2(t)V'(t, R(t)) \tag{69}$$

the potential function $V(t, r)$ being obtained by integration of Eq. 40. Assuming, for example, that $M = 7.2$ and $\nu = 0$ a numerical integration yields the following maximum and minimum values of R:

$$R_{min} = 2.5, \quad R_{max} = 2.8 \tag{70}$$

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The maximum value of the central density corresponding to the assumptions made above is:

$$(\rho_0)_{max} = \rho_0(t_0 + Period/2) = 5.5 \tag{71}$$

Remark.- In this example the total mass M compatible with the values 66 has to be less than ≈ 13.

To see a physical model corresponding to the above parameters we have to choose a value of the central density at time t_0 and also a value at the same time for the temperature. Let us assume as an example that:

$$\rho_0(t_0) = 0.5 \, \text{kg/m}^3 \quad Temp(t_0) = 10^7 \, \text{K} \tag{72}$$

This corresponds to the following values:

$$\bar{T} = 8.5 \times 10^4 \, \text{s}, \quad \bar{L} = 4.2 \times 10^8 \, \text{m}, \quad \bar{M} = 1.2 \times 10^{25} \, \text{kg} \tag{73}$$

From 68, 70 and 71 we have then:

$$Period = 5.4 \times \bar{T} \, \text{s} = 4.6 \times 10^5 \, \text{s} \quad (\rho_0)_{max} = 0.9 \, \text{kg/m}^3 \tag{74}$$

and:

$$R_{min} = 2.5 \times \bar{L} \, \text{m} = 1.1 \times 10^9 \, \text{m}, \quad R_{max} = 2.8 \times \bar{L} \, \text{m} = 1.2 \times 10^9 \, \text{m}, \tag{75}$$

$$Mass = 7.2 \times \bar{M} = 9.0 \times 10^{25} \, \text{kg} \tag{76}$$

Our second example assumes that:

$$b = -4, \quad c_1 = 0 \tag{77}$$

This corresponds to a case of maximum relaxation of type II2, and most of its physical behavior is explicit in 52 and 55. In this case also the function $\mu(t)$ has a maximum (T(t) has minimum). Let us assume that t_0 is the value of the time when this maximum is reached, then from 55 we obtain that the time interval that it takes for the function $\mu(t)$ to become zero is:

$$t_{collapse} = 1.1 \tag{78}$$

If we assume for instance that:

$$\rho_0(t_0) = 20 \times 10^3 \, \text{kg/m}^3 \tag{79}$$

we obtain:

$$t_{collapse} = 465 \, \text{s} \tag{80}$$

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References

[1] Ll. Bel, *Astrophys. J.*, **155**, 83 (1969)

[2] S. Chandrasekhar *Stellar Structure*, Chap. IV, §22 Dover (1957); *Principles of Stellar Dynamics*, Chap. V, §5.8 Dover (1960)